Automorphisms of Engel Manifolds

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What are Engel manifolds?

Definition

Let E be a 4-manifold. An *Engel structure* on E is a completely nonintegrable distribution whose rank 2.

F. Engel (1889,Darboux's theorem for Engel manifolds)

Let $(E, \mathcal{D})$ be an Engel manifold, and let $p \in E$. Then, there exists a chart $(U; x, y, z, w)$ with $p \in U$ such that
$\mathcal{D}|_U = Ker(dy - zdx) \cap Ker(dz - wdx)$.

What are Engel manifolds?

R. Montgomery(1993)

A germ of rank k "good" distribution on a n-manifold is "stable"
$\Rightarrow dim(G_{k,n}) = k(n-k) \leq n$
$\Leftrightarrow k = 1$,
or, $k = n - 1$,
or, $(n, k) = (4, 2)$
or, tirivial (i.e. k or n is 0)

- The word "good" means that it admits a k-frame generating a finie dimensional Lie algebra.
- The word "stable" means that the distribution as section of a Grassmannian bundle is stable.

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$\Rightarrow dim(G_{k,n}) = k(n-k) \leq n$
$\Leftrightarrow k = 1$,
or, $k = n - 1$, $\leftarrow$contact, even contact
or, $(n, k) = (4, 2)$ $\leftarrow$Engel
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- The word "good" means that it admits a k-frame generating a finie dimensional Lie algebra.
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Previous Work

Theorem (1999, R. Montgomery)

There is an Engel manifold such that the automorphism group is 1 dimensional at most if it is a Lie group.

Question (ref. AIM Problem Lists)

Is there an Engel manifold with trivial automorphism group?

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$\longrightarrow$ The answer is “Yes”.(Mitsumatsu, Y.)

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$\longrightarrow$ The answer is "Yes".(Mitsumatsu, Y.)

However, the following question is open.

Question (Mitsumatsu)

Is there an closed Engel manifold with trivial automorphism group?

1. Introduction

2. Engel structures and contact structures
   - from Engel to contact
   - from contact to Engel

3. Main results
   - extend to orbifolds
   - the development map and Engel automorphisms
   - construction of the answer of AIM's problem

Engel structures

Definition 1.1

Let E be a 4-manifold. An *Engel structure* on E is a smooth rank 2 distribution $\mathcal{D} \subset TE$ with following condition:

$$\mathcal{D}^2 :\overset{\text{def}}{=} \mathcal{D} + [\mathcal{D}, \mathcal{D}] \text{ has rank 3, and } \mathcal{D}^3 :\overset{\text{def}}{=} \mathcal{D}^2 + [\mathcal{D}^2, \mathcal{D}^2] \text{ has rank 4.}$$

The pair $(E, \mathcal{D})$ is called an *Engel manifold*.
Let $(E_1, \mathcal{D}_1), (E_2, \mathcal{D}_2)$ be Engel manifolds. A *Engel morphism* $f : (E_1, \mathcal{D}_1) \to (E_2, \mathcal{D}_2)$ is a local diffeomorphism $f : E_1 \to E_2$ with $df(\mathcal{D}_1) \subset \mathcal{D}_2$.

Propositoin 1.2 (R. Montgomery)

Let $(E, \mathcal{D})$ be an Engel manifold. Then, there exists a unique rank 1 distribution ${}^{\exists!}\mathcal{L} \subset \mathcal{D}$ such that $[\mathcal{L}, \mathcal{D}^2] \subset \mathcal{D}^2$.

- The above $\mathcal{L}$ is called the *characteristic foliation* of $(E, \mathcal{D})$.

Example 1.3

$E \overset{\text{def}}{=} J^2(1,1) \cong \mathbb{R}^4 \ni (x, y, \dot{y}, \ddot{y})$
$\mathcal{D} \overset{\text{def}}{=} Ker(\alpha) \cap Ker(\beta)\,(\alpha \overset{\text{def}}{=} dy - \dot{y}dx, \beta \overset{\text{def}}{=} d\dot{y} - \ddot{y}dx)$
$\longrightarrow (E, \mathcal{D})$ is an Engel manifold.
Then,
$\mathcal{L} = \langle \partial_{\ddot{y}} \rangle = Ker(\alpha) \cap Ker(\beta) \cap Ker(dx)$ is the characteristic foliation.
And,
$\mathcal{D} = \langle \partial_{\ddot{y}}, X \rangle = Ker(\alpha) \cap Ker(\beta)\,(X \overset{\text{def}}{=} \partial_x + \dot{y}\partial_y + \ddot{y}\partial_{\dot{y}})$,
$\mathcal{E} \overset{\text{def}}{=} \mathcal{D}^2 = \langle \partial_{\ddot{y}}, X, \partial_{\dot{y}} \rangle = Ker(\alpha)$.

from Engel to contact

Any 3-dimensional submanifold $M \subset E$ intersecting transversally the characteristic foliation has a contact structure $TM \cap \mathcal{D}^2$.

Definition 1.4

Let M be an odd dimensional manifold. A *contact structure* on M is a corank 1 distribution $\xi \subset TM$ such that, for any local differential form α with $\xi = Ker(\alpha)$, $d\alpha|_\xi : \xi \otimes \xi \to \mathbb{R}$ is nondegenerate.
Then, the pair (M, ξ) is called a *contact manifold*. and α is called a *contact form*.
Let $(M_1, \xi_1), (M_2, \xi_2)$ be contact manifolds. A *contact morphism* $f : (M_1, \xi_1) \to (M_2, \xi_2)$ is a local diffeomorphism $f : M_1 \to M_2$ with $df(\xi_1) \subset \xi_2$.

foliations, holonomies and leaf spaces

Any vector field tangent to the characteristic foliation $\mathcal{L}$ preserves the "even contact structure" $\mathcal{D}^2$.
In particular, any holonomy of $\mathcal{L}$ is a germ of contact morphism.
So the leaf space $E/\mathcal{L}$ has a contact structure $\mathcal{D}^2/\mathcal{L}$.
We want to research the relation between an Engel structure and the contact structure, but the leaf space is not necessarily a manifold. (In general, it is a Lie groupoid.)

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Theorem 1.5

Let X be a manifold and let $\mathcal{F}$ be a foliation on X.

1. *If all leaves of $\mathcal{F}$ is compact and all holonomy groups of that are finite, then the leaf space $X/\mathcal{F}$ is an orbifold.*
2. *As above, if all holonomy groups are trivial, then the leaf space $X/\mathcal{F}$ is a manifold.*

Definition 1.6 (Y.)

Let $(E, \mathcal{D})$ be an Engel manifold, and let $\mathcal{L}$ be the characteristic foliation of $(E, \mathcal{D})$.

1. If all leaves of $\mathcal{L}$ is compact and all holonomy groups of that are finite, then we says that $(E, \mathcal{D})$ has the *proper* characteristic foliation.
2. As above, if all holonomy groups are trivial, then we says that $(E, \mathcal{D})$ has the *trivial* characteristic foliation.

Propositoin 1.7 (R. Montgomery)

Let $(E, \mathcal{D})$ be an Engel manifold that has the trivial characteristic foliation, let M be the leaf space of the characteristic foliation, and let $\pi : E \to M$ be the quotient map. Then, $\xi \overset{\mathrm{def}}{=} d\pi(\mathcal{D}^2)$ is well-defined, and it is a contact structure on M. Moreover, $(E, \mathcal{D}) \mapsto (M, \xi)$ is functorial.

Let (M,ξ) be a contact 3-manifold, let $E = \mathbb{P}(\xi) \stackrel{\text{def}}{=} \coprod_{x\in M} \mathbb{P}(\xi_x) = (\xi - 0)/\mathbb{R}^{\times}$, and let $\pi : E \to M$ is the projective map. Now, we define a rank 2 distribution $\mathcal{D}$ on E in the following way: For each $l \in E$ with $\pi(l) = x$, $l \subset \mathbb{P}(\xi_x)$ is a line that cross the origin. By the way, we define $\mathcal{D}_l \stackrel{\text{def}}{=} d\pi_l^{-1}(l) \subset T_l E$.

Similarly, let $E' = \mathbb{S}(\xi) \stackrel{\text{def}}{=} (\xi - 0)/\mathbb{R}_{>0}$. Then E' is a 2-covering on E. Now, we define a rank 2 distribution $\mathcal{D}'$ on E' as the pull-back of $\mathcal{D}$.

Propositoin 1.8 (R. Montgomery)

The above $\mathcal{D}, \mathcal{D}'$ is an Engel structure on E, E'. Moreover, $(M,\xi) \mapsto (E,\mathcal{D})$ and $(E',\mathcal{D}')$ are functorial.

the Cartan prolongation

Definition 1.9

The above $(E, \mathcal{D})$ is called *Cartan prolongation*, and we denote this $\mathbb{P}(M, \xi)$.
The above $(E', \mathcal{D}')$ is called *oriented Cartan prolongation*, and we denote this $\mathbb{S}(M, \xi)$.

Any Cartan prolongation has the trivial characteristic foliation. In fact, the Cartan prolongation is the "minimal" object of such Engel manifolds.

the development map

Let $(E, \mathcal{D})$ be an Engel manifold that has the trivial characteristic foliation, let (M, ξ) be the leaf space of the characteristic foliation, and let $\pi : E \to M$ be the quotient map. Now, we define $\phi : E \to \mathbb{P}(\xi)$ as $\phi(e) \stackrel{\mathrm{def}}{=} d\pi(\mathcal{D}_e) \subset \xi_{\pi(e)}$.

Propositoin 1.10 (R. Montgomery)

The above ϕ is an Engel morphism $(E, \mathcal{D}) \to \mathbb{P}(M, \xi)$. Moreover, this satisfies the universality: For any contact 3-manifold (N, ν) and any Engel morphism $\psi : (E, \mathcal{D}) \to \mathbb{P}(N, \nu)$, there exists a unique contact morphism $\tilde{\psi} : (M, \xi) \to (N, \nu)$ such that $\psi = \mathbb{P}(\tilde{\psi}) \circ \phi$.

- The above ϕ is called the *development map* associated to $(E, \mathcal{D})$.
- The functor $\mathbb{P}$ is fully faithful.

extend to orbifolds

The above discussion can be generalized to an Engel manifold with proper characteristic foliation in the obvious way. That is, if $E/\mathcal{L}$ is an orbifold, then $E/\mathcal{L}$ has a contact structure, and then E has the development map. However, the Cartan prolongation of a contact 3-orbifold is an Engel orbifold in general.

Question

When is the Cartan prolongation of a contact 3-orbifold an Engel manifold?

Main result 1

Theorem 2.1 (Y.)

Let (Σ, ξ) be a contact 3-orbifold.

1. *The Cartan prolongation of (Σ, ξ) is a manifold if and only if (Σ, ξ) is positive, and $|G_x|$ is odd for all $x \in \Sigma$, where G_x is the isotropy group at x.*
2. *The oriented Cartan prolongation of (Σ, ξ) is a manifold if and only if (Σ, ξ) is positive.*

In fact, all Engel manifolds obtained as the Cartan prolongation of a "space" with contact structure are obtained as above. This can be shown by using Lie groupoid theory.

- Recall, a *manifold* is a topological space to be locally Euclidean.
- An *orbifold* is a topological space to be locally Euclidean with a finite group action.

Definition 2.2

Let Σ be an orbifold, $\{(V_\lambda \subset \mathbb{R}^n, G_\lambda \curvearrowright V_\lambda, p_\lambda : V_\lambda \to \Sigma)\}_{\lambda \in \Lambda}$ be an orbifold atlas on Σ, and $\phi_{\lambda\mu}$ be the transformation map for $\lambda, \mu \in \Lambda$.
A *contact structure* on Σ is a family $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$ of contact structures on each V_λ such that all $G_\lambda \curvearrowright V_\lambda$ are contact actions, and all $\phi_{\lambda\mu}$ are contact morphisms. The pair of an orbifold and a contact structure is called a *contact orbifold*.
Similarly, we define an *Engel structure* on an orbifold and an *Engel orbifold*.

Example 2.3

$\xi_{std} \stackrel{\text{def}}{=} Ker(dz + xdy - ydx)$ is a contact structure on $\mathbb{R}^3$. We define $\phi_n, \psi : \mathbb{R}^3 \to \mathbb{R}^3$ as $\phi_n(x, y, z) \stackrel{\text{def}}{=}$ $(x\cos(\frac{2\pi}{n}) - y\sin(\frac{2\pi}{n}), x\sin(\frac{2\pi}{n}) + y\cos(\frac{2\pi}{n}), z), \psi(x, y, z) \stackrel{\text{def}}{=} (x, -y, -z)$. Then, $G_{n,std} \stackrel{\text{def}}{=} < \phi_n, \psi > \curvearrowright (\mathbb{R}^3, \xi_{std})$ and $H_{n,std} \stackrel{\text{def}}{=} < \phi_n > \curvearrowright (\mathbb{R}^3, \xi_{std})$ are contact actions of finite groups. These are called *standard models*. Then, $(\mathbb{R}^3, \xi_{std})/G_{n,std}$ and $(\mathbb{R}^3, \xi_{std})/H_{n,std}$ are contact orbifolds.

Theorem 2.4 (Darboux's theorem for contact 3-orbifolds)

Let (Σ, ξ) be a contact 3-orbifold, and let $x \in \Sigma$. Then, there exists an orbifold chart (V, G, p) around x such that (V, G) is isomorphic to an open neighborhood of $0 \in (\mathbb{R}^3, H)$ where $H = G_{n,std}$ or $H = H_{n,std}$.

isotoropies and positive contact orbifoflds

Definition 2.5

Let Σ be an orbifold and $x \in \Sigma$. Take an orbifold chart (V, G, p) with $x \in p(V) \subset \Sigma$, and take a point $\tilde{x} \in p^{-1}(x)$. Then, $G_x \overset{\mathrm{def}}{=} \{\sigma \in G \mid \sigma\tilde{x} = \tilde{x}\}$ is called the *isotropy group* at x.

Definition 2.6

A contact orbifold (Σ, ξ) is *positive* if all $G_x \curvearrowright \xi_x$ preserve the orientation.

- $(\mathbb{R}^3, \xi_{std})/G_{n,std}$ is not positive.
- $(\mathbb{R}^3, \xi_{std})/H_{n,std}$ is positive.

Main lemma

Because of Darboux's theorem for contact 3-orbifolds, the main result follows the following lemma.

Lemma 2.7

1. *The actions* $G_{n,std} \curvearrowright \mathbb{P}(\xi_{std})$, $G_{n,std} \curvearrowright \mathbb{S}(\xi_{std})$ *are not free.*
2. *The action* $H_{n,std} \curvearrowright \mathbb{P}(\xi_{std})$ *is free if and only if* n *is odd.*
3. *The action* $H_{n,std} \curvearrowright \mathbb{S}(\xi_{std})$ *is free for any* n*.*

Main result 2

The composition of the two functors

$$\begin{array}{ccccc} \{Engel\} & \longrightarrow & \{Contact\} & \longrightarrow & \{Engel\} \\ \Downarrow & & \Downarrow & & \Downarrow \\ (E,\mathcal{D}) & \longmapsto & (E/\mathcal{L},\mathcal{D}^2/\mathcal{L}) & \longmapsto & \mathbb{P}(E/\mathcal{L},\mathcal{D}^2/\mathcal{L}) \end{array}$$

induces a group morphism

$$\Phi : Aut(E,\mathcal{D}) \to Aut(E/\mathcal{L},\mathcal{D}^2/\mathcal{L}) \cong Aut(\mathbb{P}(E/\mathcal{L},\mathcal{D}^2/\mathcal{L}))$$

Theorem 2.8 (Y.)

Let $(E,\mathcal{D})$ be a connected Engel manifold. Suppose that $\mathbb{P}(E/\mathcal{L},\mathcal{D}^2/\mathcal{L})$ is a manifold. If the development map $\phi : (E,\mathcal{D}) \to \mathbb{P}(E/\mathcal{L},\mathcal{D}^2/\mathcal{L})$ is not covering, then the above group morphism Φ is injective.

Main lemma

Lemma 2.9

Let $(E, \mathcal{D})$ be an Engel manifold. Suppose that $\mathbb{P}(E/\mathcal{L}, \mathcal{D}^2/\mathcal{L})$ is a manifold.

1. *If there exists a leaf L of $\mathcal{L}$ such that $\phi|_L : L \to \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$ is not covering, then the above group morphism Φ is injective.*
2. *If, for any leaf L of $\mathcal{L}$, $\phi|_L : L \to \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$ is covering, then ϕ is covering*

outline of proof

proof of 1.

Suppose $\Phi(f) = \Phi(g)$ for $f, g \in Aut(E, \mathcal{D})$.
Because $\phi|_L$ is not covering, we can identify $\phi|_L$ with $(a, b) \to \mathbb{R}/\mathbb{Z}\,(a \neq -\infty)$.

$$\begin{array}{ccc}
(c, d) & \xrightarrow{\ \phi\ } & \mathbb{R}/\mathbb{Z} \\
f \Big\uparrow \Big\uparrow g & & \Big\uparrow \Phi(f)=\Phi(g) \\
(a, b) & \xrightarrow[\ \phi\ ]{} & \mathbb{R}/\mathbb{Z}
\end{array}$$

Then there exists $\epsilon > 0$ such that $f(a+\epsilon) = g(a+\epsilon)$.
Because E is connected, we obtain $f = g$.

$\square$

proof of 2.

Take any leaf L of $\mathcal{L}$, and let $L' \stackrel{\mathrm{def}}{=} \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L$.
There exists a nonsingular vector fiald X realizing the holonomy on L', defined on a open neighborhood U.
Let $\tilde{X}$ be a pull-back of X by ϕ. Because any $\phi|_{L_0}$ is covering, $\tilde{X}$ is complete.
So, we can obtain $\phi|_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ is covering whose degree is the same as that of $\phi|_L$.

□

Let $(E, \mathcal{D})$ be a connected Engel manifold. Suppose that $E/\mathcal{L}$ is a manifold.
Define $\sigma : E/\mathcal{L} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ by $\sigma(L) \overset{\text{def}}{=} \min\{\#\phi^{-1}(y) \,|\, y \in \mathbb{P}(\mathcal{D}^2/\mathcal{L})_L\}$.
Call σ *twisting number function*.

Propositoin 2.10

For any $f \in Aut(E, \mathcal{D})$, the induced automorphism $E/\mathcal{L} \to E/\mathcal{L}$ preserves the twisting number function σ.

Example 2.11

$E \overset{\text{def}}{=} \mathbb{R}^4 \ni (x, y, z, \theta)$

$\mathcal{D} \overset{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2})X + \sin(\frac{\theta}{2})Y \rangle$ $(X \overset{\text{def}}{=} \partial_x - y\partial_z, Y \overset{\text{def}}{=} \partial_y)$

$\longrightarrow$ $(E, \mathcal{D})$ is an Engel manifold.

Then, $\mathcal{L} = \langle \partial_\theta \rangle$ is the characteristic foliation.

And, $\mathcal{E} \overset{\text{def}}{=} \mathcal{D}^2 = \langle \partial_\theta, X, Y \rangle$.

$\longrightarrow M \overset{\text{def}}{=} E/\mathcal{L} \cong \mathbb{R}^3 \ni (x, y, z)$, $\xi \overset{\text{def}}{=} \mathcal{E}/\mathcal{L} \cong \langle X, Y \rangle$.

$M \times S^1 \cong \mathbb{P}(M, \xi); (\mathbf{x}, [\theta]) \mapsto \langle \cos(\frac{\theta}{2})X_{\mathbf{x}} + \sin(\frac{\theta}{2})Y_{\mathbf{x}} \rangle$.

$\phi : E \to M \times S^1; (\mathbf{x}, \theta) \mapsto (\mathbf{x}, [\theta])$ is identified with the development map.

$\longrightarrow$ The twisting number function σ is a constant function ∞.

Example 2.12

Fix a point $\mathbf{x}_0 \in \mathbb{R}^3$ and a number $n \in \mathbb{Z}_{\geq 0}$.

$E \overset{\text{def}}{=} \mathbb{R}^4 - \{\mathbf{x}_0\} \times ((-\infty, -n\pi] \cup [n\pi + \epsilon, \infty))\,(\epsilon \in (0, 2\pi])$

$\mathcal{D} \overset{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2})X + \sin(\frac{\theta}{2})Y \rangle\,(X \overset{\text{def}}{=} \partial_x - y\partial_z, Y \overset{\text{def}}{=} \partial_y)$

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$\longrightarrow M \overset{\text{def}}{=} E/\mathcal{L} \cong \mathbb{R}^3 \ni (x, y, z)$, $\xi \overset{\text{def}}{=} \mathcal{E}/\mathcal{L} \cong \langle X, Y \rangle$.

Then, $M \times S^1 \cong \mathbb{P}(M, \xi); (\mathbf{x}, [\theta]) \mapsto \langle \cos(\frac{\theta}{2})X_{\mathbf{x}} + \sin(\frac{\theta}{2})Y_{\mathbf{x}} \rangle$.

$\phi : E \to M \times S^1; (\mathbf{x}, \theta) \mapsto (\mathbf{x}, [\theta])$ is identified with the development map.

$\longrightarrow$ The twisting number function σ is the following:

$$\sigma(\mathbf{x}) = \begin{cases} n & (\mathbf{x} = \mathbf{x}_0) \\ \infty & (\textit{otherwise}) \end{cases}$$

Theorem 2.13 (Mitsumatsu, Y.)

There exists an Engel manifold with trivial automorphism group.

proof

Take a countable dense subset $Q = \{\mathbf{x}_n\}_{n=1}^{\infty} \subset \mathbb{R}^3$.

$E \overset{\text{def}}{=} \mathbb{R}^4 - \bigcup_n \{\mathbf{x}_n\} \times ((-\infty, -n\pi] \cup [n\pi + \epsilon, \infty))\, (\epsilon \in (0, 2\pi])$

$\mathcal{D} \overset{\text{def}}{=} \langle \partial_\theta, \cos(\frac{\theta}{2})X + \sin(\frac{\theta}{2})Y \rangle\, (X \overset{\text{def}}{=} \partial_x - y\partial_z, Y \overset{\text{def}}{=} \partial_y)$

$\longrightarrow$ $(E, \mathcal{D})$ is an Engel manifold.

$\longrightarrow$ The twisting number function σ is the following:

$$\sigma(\mathbf{x}) = \begin{cases} n & (\mathbf{x} = \mathbf{x}_n) \\ \infty & (\mathit{otherwise}) \end{cases}$$

For any $f \in Aut(E, \mathcal{D})$, the induced automorphism $\underline{f} : \mathbb{R}^3 \to \mathbb{R}^3$ is the identity on Q. Because $Q \subset \mathbb{R}^3$ is dense, $\underline{f}$ is the identity on $\mathbb{R}^3$. So f is the identity.

□